DEVELOPMENT OF A PLANE CRACK UNDER THE ACTION OF A SUDDENLY APPLIED UNIFORM PRESSURE

V. A. Saraikin

UDC 539.30

The growth of a crack is considered in the case of a suddenly applied uniformly distributed pressure on the banks. This plane problem is a particular case of the problem concerned with crack propagation with an arbitrary variable velocity [1-4], since the law of motion of the crack here is not arbitrary, but is determined from the fracture criterion for dynamics [5, 6].

With the assumption that the specific surface energy is unaltered, in the paper the law of motion of a semiinfinite crack developing in a brittle material is calculated. Side by side with the exact solution of the theory of elasticity, for comparison purposes, relations are constructed which describe the same process approximately (for an approximate model of the crack see [3, 4, 7, 8]). Over the extent of the crack stress oscillograms determined by the theory of elasticity and by the approximate model are compared. We note that crack development in the case of antiplane strain is investigated in [9].

Let the position of the crack at the time instant t = 0 be specified in the Cartesian system of coordinated as follows: $-l_0^- < x < l_0^+,\ -\infty < y < \infty,\ z = 0\ \ (l_0^{\pm} > 0)$. The source of perturbations is uniform pressure applied suddenly to the banks of the crack at t = 0. The tangential stress is zero. The crack tips for $t > 0$ are displaced: to the right according to the law $x = l^+(t)$, to the left according to the law $x = -l^-(t)$, with $l^{\pm}(0) = l_0^{\pm}$. At the same time the stresses indicated above arise also on the newly formed surfaces of the crack. Because of symmetry of the load about the plane of the crack over its continuation, tangential stresses and displacements of points along the z axis are absent. The initial conditions of the problem are zero conditions.

We consider the motion of the right end of the crack up to the time instant corresponding to the arrival of perturbations from the left boundary. In this time interval the solution of the problem for $x > l(t)$ (the index + is dropped) coincides with the solution for a semi-infinite crack on the banks of which there specified a normal and a tangential component of the stress tensor

$$\begin{gathered}\sigma_{zz} = \sigma_- = -p_0 H[l(t) - x]H(t),\\ \sigma_{xz} = 0 \quad (z = \pm 0),\end{gathered} \tag{1}$$

where H is the Heaviside function, and p_0 is constant with dimensions.

We write out the solution of the problem (1), having made use of the results of [3, 8]. Assuming the velocity of motion of the crack tip to be not negative and bounded from above by the velocity of Rayleigh waves, the normal stress σ_{zz} on the continuation of the crack after simple transformations can be represented by the following expressions:

$$\begin{gathered}\sigma_+ = \frac{2}{\pi}\left[AN(t, x, a) + \int_a^b F_3(h) N(t, x, h)\, dh\right] H\left[t - a\left(x - l_0^+\right)\right],\\ N(t, x, h) = p_0\left[\frac{a(1 - cl^{\cdot})}{c(1 - hl^{\cdot})}\sqrt{\frac{t_0}{a(x - l)}}\, D + \sqrt{\frac{a}{h}} \operatorname{arctg}\sqrt{\frac{t - t_0}{t_0}}\, D - \right.\\ \left. - \frac{\pi}{2}\left(1 + \int_{u_1}^b F_1(u)\sqrt{\frac{u - a}{u - h}}\frac{du}{u}\right)\right] H\left[t - h\left(x - l_0^+\right)\right],\\ A = 1 + \int_a^b F_2(u)\frac{du}{u - a},\ \ D = 1 + \int_a^b F_1(u)\frac{du}{u},\ \ u_1 = \frac{t}{x - l + t_0/b},\\ l = l(t_0),\ \ l^{\cdot} = dl(t_0)/dt_0,\quad t - t_0 = h\left[x - l(t_0)\right],\end{gathered} \tag{2}$$

where a, b, c are quantities reciprocal to the velocities of dilatation, shear, and Rayleigh; the functions F_i, not depending on the law of motion and the load, have the form [8]

$$\begin{gathered}F_i = (-1)^{i+1}\gamma(u)\exp\left[(-1)^{i+1}\varkappa(u)\right]\ (i = 1, 2),\\ \gamma(u) = \frac{4}{\pi}u^2\sqrt{b^2 - u^2}\sqrt{u^2 - a^2}\left[(b^2 - 2u^2)^4 + 16u^4(b^2 - u^2)(u^2 - a^2)\right]^{-1/2},\end{gathered}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 172-177, May-June, 1980. Original article submitted June 14, 1979.

0021-8944/80/2103-0428$07.50

$$\varkappa(u)=\frac{1}{\pi}\,\mathrm{v.p.}\int\limits_a^b \operatorname{arctg}\left[4\alpha^2\sqrt{\alpha^2-a^2}\sqrt{b^2-\alpha^2}\,(b^2-2\alpha^2)^{-2}\right]\frac{d\alpha}{\alpha-u},$$

$$F_3(h)=\int\limits_h^b \frac{d}{du}\left[\frac{F_2(u)}{\sqrt{u-a}}\right]\frac{du}{\sqrt{u-h}}.$$

The symbol v.p. denotes the principal value according to Cauchy in the case of integration.

The function $l(t)$ in (2) as yet is arbitrary, while the development of the crack is connected with the energy used at its boundary. Therefore the law of motion of the crack has to be determined from the condition of conservation of energy in the process of deformation for an elastic body with a developing crack – an analogy of the Griffiths–Irvine criterion for dynamics. If we denote by T the density of energy absorbed at the boundary of the crack, then from the law of conservation of energy for a crack of normal fracture we have the following connection of T with the dynamic intensity coefficient:

$$\begin{gathered}2T(l^{\cdot})=b^2l^{\cdot 2}\sqrt{1-a^2l^{\cdot 2}}\,K_{\mathrm{I}}^2/[2\mu R(l^{\cdot})],\\ K_{\mathrm{I}}=\lim_{x\to l(t)+0}\sqrt{2\pi(x-l)},\ \sigma_+=2p_0\sqrt{\frac{2at}{\pi c^2}}\,DK_1(l^{\cdot}),\\ K_1(l^{\cdot})=\frac{1-cl^{\cdot}}{\sqrt{1-al^{\cdot}}}\,k(l^{\cdot}),\quad k(l^{\cdot})=1-l^{\cdot}\int\limits_a^b F_2(u)\frac{du}{1-ul^{\cdot}},\\ R(l^{\cdot})=4\sqrt{1-a^2l^{\cdot 2}}\sqrt{1-b^2l^{\cdot 2}}-(2-b^2l^{\cdot 2})^2,\quad l=l(t)\end{gathered}\tag{3}$$

(μ is the shear modulus).

The graphs of the function $K_1(l^{\cdot})$ are shown in Fig. 1, where the curves 1-3 correspond to the values of Poisson's ratio $\nu = 0.2; 0.25; 0.3$. This function coincides with the coefficient of dynamism $K_1(\nu)$ of [2] (the expressions (2.11), (2.12)).

The relation (3) serves as the equation for determining the velocity of the crack. The crack is stationary ($l = l_0^+$, $l^{\cdot} = 0$) as long as $t < t_s$. To the quantity t_s corresponds the time when the dynamic coefficient of intensity, growing, reaches the value of the static coefficient of intensity $K_{\mathrm{I}s}$ for the crack in limit equilibrium

$$K_{\mathrm{I}}^2=K_{\mathrm{I}s}^2=2ET(0)/(1-\nu^2)\tag{4}$$

(the constant E is Young's modulus). Hence we find the time of start of the motion of the right boundary of the crack

$$t_s=\frac{\pi c^2 ET(0)}{4p_0^2a(1-\nu^2)D^2}.\tag{5}$$

The law of motion of the crack for $t > t_s$ is determined, taking into account (3), (5), from the equation

$$\begin{gathered}f(l^{\cdot})t/t_s=T(l^{\cdot})/T(0),\\ f(l^{\cdot})=\sqrt{\frac{1+al^{\cdot}}{1-al^{\cdot}}}\,\frac{[bl^{\cdot}(1-cl^{\cdot})\,k(l^{\cdot})]^2}{(1-\nu)\,R(l^{\cdot})}.\end{gathered}\tag{6}$$

As the initial condition serves the position of the crack tip at the time instant $t = t_s$.

In the calculations it was assumed that the specific surface energy does not depend on the velocity of motion of the crack $T(l^{\cdot}) = T(0) = \text{const}$. As the unit of measurement of the length and velocity we took l_0^+ and the velocity of the shear wave. The density of the medium was also equal to unity. In addition to this, in the solution the following quantities have been specified:

$$\nu=0.3,\ ET(0)/[p_0^2(1-\nu^2)\,l_0^+]=1.$$

The motion of the crack with such a choice of the parameters commences at the time instant $t_s/(bl_0^+) = 1.0286$.

In Fig. 2, curve 1 represents the relation $x = l(t)$. On the abscissa axis we have marked off the quantity x/l_0^+, while on the ordinate axis we have marked off $-t/(bl_0^+)$. The variation of the velocity of the crack with time in the expression (6) is shown by curve 1 in Fig. 3. Curve 1 presented in Fig. 4 gives an idea about the variation with time of the stress σ_+ at the point $x = 2l_0^+$. On the abscissa axis we have plotted the quantity $t/(bl_0^+)$, and the ordinate axis is the dimensionless stress σ_+/p_0. Up to the time instant $t = a(x + l_0^-)$ the stress at this point, calculated for a semiinfinite crack, coincides with the stress radiated onto the continuation by the crack $-l^-(t) < x < l^+(t)$. Afterwards at the point under consideration there arrive perturbations from the left boundary of the crack, and the solution must be constructed with reflections of waves from the right boundary of the crack taken into account.

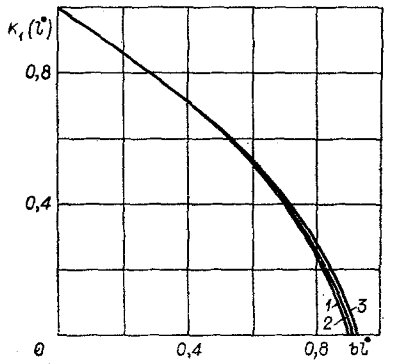

Fig. 1

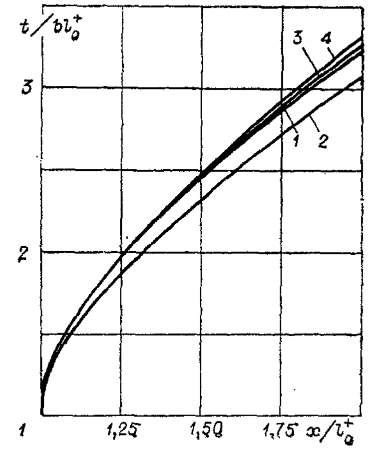

Fig. 2

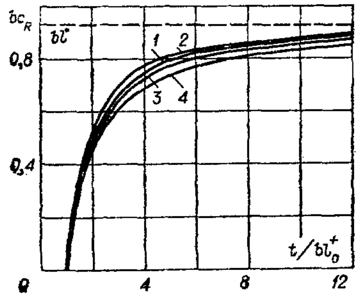

Fig. 3

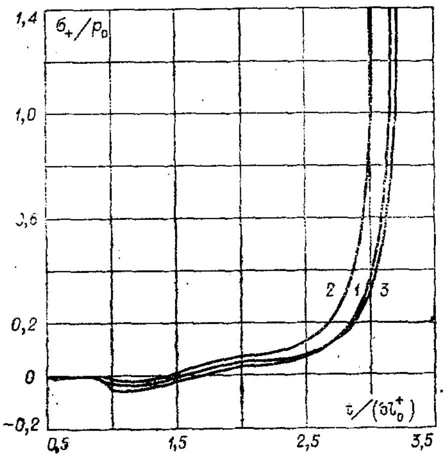

Fig. 4

We examine what should the pressure p_0 be for the motion of the right boundary of the crack to commence earlier than perturbations arrive from its left boundary. From the condition $t_s < a(l_0^+ + l_0^-)$ we find, with (5) taken into account, the inequality

$$p_0 > (\pi c p_s)/(4aD),$$

where p_s is the limit static pressure for the crack $-l_0^- < x < l_0^+$ determined from the solution of the static problem and from (4)

$$K_{Is}^2 = \pi p_s^2 (l_0^+ + l_0^-)/2 = 2ET(0)/(1-\nu^2).$$

Since in the calculations we have taken $ET(0)/[p_0^2(1-\nu^2)l_0^+] = \alpha = \mathrm{const}$, we find that in this case

$$p_0^2 = \pi p_s^2 (1 + l_0^-/l_0^+)/(4\alpha) > [(\pi c p_s)/(4aD)]^2$$

and, consequently, it must be

$$l_0^-/l_0^+ > [\pi c^2 \alpha/(4a^2D^2)] - 1.$$

Increasing still more the value p_0 (at the expense of l_0^-), we can achieve that the boundary of the crack arrives at the point $x = 2l_0^+$ (being considered in the example) earlier than the perturbations radiated by the left end of the crack.

In Figs. 2-4 we have depicted also the solutions obtained for $\nu = 0.3$ from the approximate model [3]

$$\sigma_+ = \frac{2p_0}{\pi}\left[\frac{a_*(1-c\dot{l})}{c(1-a_*\dot{l})}\sqrt{\frac{t^*}{a_*(x-l)}} - \operatorname{arctg}\sqrt{\frac{t^*}{t-t^*}}\right] H\left(\frac{t}{x-l_0^+} - a_*\right),$$

$$t - t^* = a_*[x - l(t^*)].$$

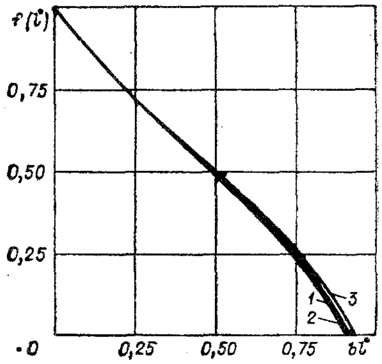

Fig. 5

The choice of values of the parameter a_* in the approximate model is discussed in [8]. For $\nu = 0.3$ we have $a_* = a_1 = 0.9469$ [7] (curve 2), $a_* = a_2 = 0.8879$ [8] (curve 3).

The law of motion of the crack in the calculations as before was found from the condition (3); only instead of K_I its approximate value was substituted

$$K_I^* = 2p_0\sqrt{\frac{2a_*t}{\pi c^2}}\frac{1-cl^\cdot}{\sqrt{1-a_*l^\cdot}}, \quad l = l(t).$$

As a result, the law of motion of the crack is determined from the equation

$$f^*(l^\cdot)\,t/t_s^* = T(l^\cdot)/T(0),$$

$$f^*(l^\cdot) = \frac{\sqrt{1-a^2l^{\cdot 2}}\,[bl^\cdot(1-cl^\cdot)]^2}{(1-a_*l^\cdot)(1-\nu)R(l^\cdot)}, \quad t_s^* = \frac{\pi c^2 ET(0)}{4p_0^2(1-\nu^2)a_*},$$

where t_s^* is the time of the start of motion of the crack. In the example being considered, for $a_* = a$, it practically coincides with t_s, while for $a_* = a_1$ it is less: $t_s^*/(bl_0^+) = 0.9626$.

Calculations show that the approximation solution for $a_* = a_2$ better than for $a_* = a_1$ approximates the relations determined by the theory of elasticity. Comparisons carried out in the investigation allow us to draw the conclusion about the applicability of the approximate model for the calculation of problems concerned with dynamic fracture.

When determining the law of motion of the crack from Eq. (6), it turned out that the function $f(l^*)$ in the interval $0 \leqslant bl^* < 0.4$ only insignificantly differs from a linear function. In Fig. 5 curves 1-3 show its variation for $\nu = 0.2; 0.25; 0.3$.

We replace the function on this interval by the approximate expression

$$f(l^\cdot) \sim 1 - cl^\cdot(t). \tag{7}$$

Integrating Eq. (6), where $f(l^\cdot)$ is replaced by the function (7), we find the law of motion of the crack under a load uniformly distributed over the banks

$$l(t) = l_0^+ + [t - t_s - t_s\ln(t/t_s)]/c. \tag{8}$$

The values of l and $l^\cdot$ calculated from the expression (8) for $\nu = 0.3$ are shown in Figs. 3 and 4 (curves 4). They quite well approximate the exact solution at the start of the fracture process.

LITERATURE CITED

1. L. B. Freund, "Crack propagation in an elastic solid subjected to general loading," J. Mech. Phys. Solids, 20, No. 3 (1972).
2. B. V. Kostrov, "Crack propagation at a variable velocity," Prikl. Mat. Mekh., 38, No. 3 (1974).
3. L. I. Slepyan, "An approximate model of crack dynamics," in: Dynamics of Solid Media [in Russian], No. 19-20, Inst. Gidrodin. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1974).
4. L. I. Slepyan and L. V. Troyankina, The Theory of Cracks. Basic Ideas and Results [in Russian], Sudostroenie, Leningrad (1976).
5. B. V. Kostrov, L. V. Nikitin, and L. M. Flitman, "Mechanics of brittle fracture," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3 (1969).
6. E. N. Sher, "On the energy condition at the tip of a nonstationary crack," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1969).
7. L. R. F. Rose, "An approximate (Wiener–Hopf) kernel for dynamic crack problems in linear elasticity and viscoelasticity," Proc. R. Soc., Ser. A, 349, No. 1659 (1976).
8. V. A. Saraikin and L. I. Slepyan, "A plane problem of crack dynamics in an elastic body," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 4 (1979).
9. A. E. Molchanov and L. V. Nikitin, "Dynamics of a crack of longitudinal shear after loss of stability," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 2 (1972).